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Physical flow properties for Pollard-like internal water waves

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We present a study of the physical flow properties for a recently derived three-dimensional nonlinear geophysical internal wave solution. The Pollard-like internal wave solution is explicit in terms of Lagrangian labelling variables, enabling us to examine the mean flow velocities and mass flux in the three-dimensional setting. We show that the Pollard-like internal water wave does not have a net wave transport. *Published by AIP Publishing.* https://doi.org/10.1063/1.5038657

I. INTRODUCTION

This paper is devoted to an analysis of the mean flow properties for a recently derived[29] nonlinear geophysical internal water wave solution. In Ref. 29, the derived exact and explicit solution for internal water waves is a significant modification of the solution given by Pollard for surface waves.[32] The Pollard-like internal wave solution is explicit in terms of Lagrangian labelling variables, enabling us to examine the mean flow properties and mass transport, which in the case of the Pollard-like solution are three-dimensional vectors. Since particle paths in the internal water wave solution are circles tilted toward the equator (cf. Fig. 1), we must distinguish the flow properties in the zonal, meridional, and vertical direction.[4]

The Gerstner-like solution represents a two-dimensional periodic traveling water wave with the particle paths in the vertical plane,[3,4,13,15,18] whereas in the Pollard-like solution, waves experience a slight cross-wave tilt.[10,29,32] Various Gerstner-type solutions of the geophysical fluid dynamic equations in the equatorial region have been derived, in particular, to describe nonlinear internal water waves[6,7] and wave-current interactions.[16,17,27,28,34] An analysis of the mean physical flow properties for those equatorially trapped waves was presented in Refs. 20 and 36. A survey of this extensive research activity concerning the generalised Gerstner-like solutions was summarised in Refs. 19, 24, and 26, and a discussion on the oceanographical relevance of those solutions can be found in Ref. 2.

The aim of this paper is to present an analysis of the physical flow properties for internal water waves. The internal water waves may describe the oscillation of a thermocline,[8,11,38] which is an interface separating two layers of ocean water of different but constant densities in a stable stratification;[11,12,38] we remark that the mechanism of generation of the oscillation of the thermocline is out of the scope of this paper. The waves propagate over the thermocline with the amplitude decreasing with height above the thermocline. The study of the thermocline and its interaction with surface waves and the Equatorial Undercurrent can be found in Refs. 8 and 25.

The Pollard-like solution is an extension of the remarkable Gerstner's solution by including the effects of the rotation of earth.[32] In the paper,[10] a Pollard-like solution was constructed to describe the wave-current interactions for surface waves. Moreover, the surface wave solution was subjected to a stability analysis[22–24] and it was proved by degree-theoretic methods to be globally dynamically possible.[35] Subsequently, in a recent study by the author, a new Pollard-like solution was constructed to model the internal water waves.[29] We use the newly developed Pollard-like solution for internal water waves to examine the mean physical flow properties, which serves as an opportunity to develop an understanding of the nature of oceanic flows on earth.[9]

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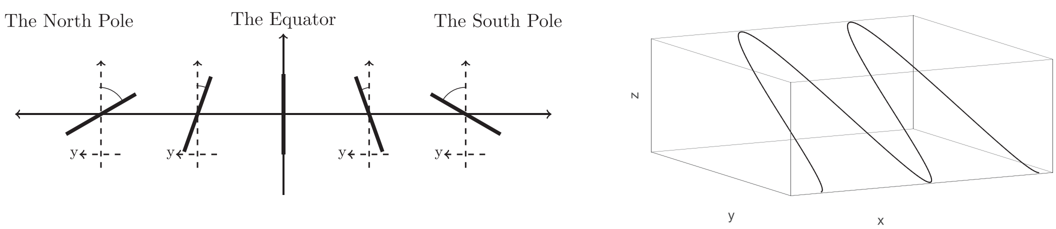

FIG. 1. (a) The figure of the inclination of the particle orbits with an angle of inclination arctan($-d/a$) increasing with the latitude, resulting in the three-dimensional profile of the internal water wave.[29] (b) The schematic 3D profile of the internal water wave outside the equatorial region.

II. THE INTERNAL WATER WAVE—THE POLLARD-LIKE SOLUTION

A. Governing equations

We consider the geophysical internal water waves, where we assume that earth is a perfect sphere. The frame of reference is rotating with earth whose origin is fixed at a point on earth. In this case, the (x, y, z) Cartesian coordinates represent the directions of the longitude, latitude, and local vertical, respectively. The governing equations for the geophysical water waves are given by[11]

$$\begin{cases} u_t + uu_x + vu_y + wu_z + 2\Omega w\cos\phi - 2\Omega v\sin\phi = -\frac{1}{\rho}P_x, \\ v_t + uv_x + vv_y + wv_z + 2\Omega u\sin\phi = -\frac{1}{\rho}P_y, \\ w_t + uw_x + vw_y + ww_z - 2\Omega u\cos\phi = -\frac{1}{\rho}P_z - g, \end{cases}$$

together with the equation of mass conservation

$$\rho_t + u\rho_x + v\rho_y + w\rho_z = 0$$

and the equation for incompressibility

$$u_x + v_y + w_z = 0. \tag{1}$$

In those equations, t stands for time, ϕ represents the latitude, (u, v, w) is the fluid velocity, $g = 9.81$ m s^{-2} is the gravitational acceleration on earth's surface, ρ is the water's density, and P is the pressure. The radius of earth is assumed to be $R = 6371$ km, and $\Omega = 7.29 \times 10^{-5}$ rad s^{-1} is the constant rotational speed of earth. Since we investigate the flow in a relatively narrow strip less then few degrees,[11] the Coriolis parameters

$$f = 2\Omega\sin\phi, \qquad \hat{f} = 2\Omega\cos\phi \tag{2}$$

can be taken as constants. The typical values of the Coriolis parameters at the latitude of 45° N are $f = \hat{f} = 10^{-4}$ s^{-1}.[14]

The solution to the governing equations constructed in Ref. 29 describes internal waves representing the oscillation of a thermocline. The thermocline is defined as an interface separating layers of ocean water of two constant but different densities.[11,12] The layer $\mathcal{M}(t)$ of less dense, warmer ocean water of density ρ_0 overlays the layer $\mathcal{S}(t)$ of colder, more dense water with density $\rho_+ > \rho_0$. The thermocline is denoted by $z = \eta(x, y, t)$, where there is a dramatic jump in the density of water. The oscillation of the thermocline propagates in the layer $\mathcal{M}(t)$. The layer $\mathcal{M}(t)$ is finite and is bounded by the interface $z = \eta_+(x, y, t)$, above which we have the near-surface layer $\mathcal{L}(t)$. We can consider the layer $\mathcal{M}(t)$ as finite since the amplitude of internal waves decreases exponentially as we ascend above the thermocline, and at the height of half a wave-length, the amplitude is 4% of its value at the thermocline (cf. Ref. 5). The geophysical internal wave motion, for the purpose of this model, can be neglected in the near-surface layer since the wave motion there is only a small perturbation of the surface generated primarily by the wind. The schematic is presented in Fig. 2. The assumption of still water under the thermocline is expressed as

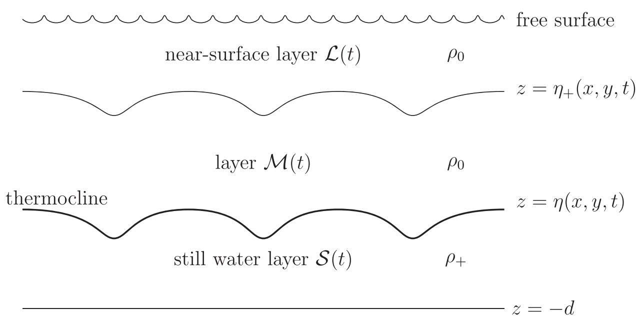

FIG. 2. The schematic showing the main regions of the flow at a fixed latitude y. The thermocline separates two layers of different densities $\rho_0 < \rho_+$ in a stable stratification. The internal wave is described by a trochoid propagating with a speed c. The amplitude of the internal water wave decreases exponentially. At the depth of half a wave-length above the thermocline, the amplitude is reaching 4% of its initial value at the thermocline.

$$(u, v, w) = (0, 0, 0) \text{ for } z < \eta(x, y, t),$$

and it implies the hydrostatic pressure in the layer $\mathcal{S}(t)$,

$$P = P_0 - \rho_+ g z \text{ for } z < \eta(x, y, t).$$

We have to emphasise that close to the thermocline, there is a region of high shear and strong stratification. Therefore, the velocity field is continuous in the normal direction at the thermocline; on the other hand, it can be or even is discontinuous in the tangential direction. The assumption of a hydrostatic still water layer $\mathcal{S}(t)$ is a rather strong assumption in this context; however, it still represents a complex (if quite simplified) model whereby a nonlinear exact internal geophysical wave solution can be constructed at mid-latitudes. It is hoped that future work may result in an exact solution for a more physically realistic nonhydrostatic layered model (as exists for internal equatorial waves in Ref. 7), yet this situation promises to be vastly more complex and technical mathematically due to the fact that flows are not considered to be purely equatorial. Nevertheless, it appears that the simplified model that we include in this paper manages to capture the salient geophysical features of flows.

In the f-plane approximation, the governing equations of the internal water waves in the layer $\mathcal{M}(t)$ are

$$\begin{cases} u_t + u u_x + v u_y + w u_z + \hat{f} w - f v = -\frac{1}{\rho_0} P_x, \\ v_t + u v_x + v v_y + w v_z + f u = -\frac{1}{\rho_0} P_y, \\ w_t + u w_x + v w_y + w w_z - \hat{f} u = -\frac{1}{\rho_0} P_z - g, \end{cases} \tag{3}$$

which are coupled with the boundary conditions

$$P = P_0 - \rho_+ g z \text{ on the thermocline } z = \eta(x, y, t),$$
$$w = \eta_t + u\eta_x + v\eta_y \text{ on the thermocline } z = \eta(x, y, t).$$

The first equation is the dynamic condition and the second is the kinematic condition. The kinematic condition prevents mixing of the particles between layers.

B. Exact and explicit solution

In this section, we briefly describe a recently constructed exact and explicit solution for internal water waves,[29] which is described in terms of Lagrangian labelling variables.[1] It represents a periodic internal wave propagating zonally with a wave speed c. The Lagrangian positions (x, y, z) of a fluid particle are given as functions of the labelling variables (q, r, s), time t and real parameters a, b, c, d, k, m. However, the parameters b, c, d must be suitably chosen in terms of k, m, a. The explicit solution to the governing equation (3) satisfying the incompressibility condition is given by[29]

$$\begin{cases} x = q - be^{-ms}\sin[k(q-ct)], \\ y = r - de^{-ms}\cos[k(q-ct)], \\ z = s - ae^{-ms}\cos[k(q-ct)]. \end{cases} \tag{4}$$

The constant $k = 2\pi/L$ is the wave number corresponding to the wavelength L, with the parameter q covering the real line. The solution is set up around a fixed latitude ϕ and therefore $r \in [-r_0, r_0]$, for some r_0. For every fixed value of $r \in [-r_0, r_0]$, we require $s \in [s_0, s_+]$, where the choice $s = s_0 \geq s^* > 0$ represents the thermocline $z = \eta(x, y, t)$ at the fixed latitude, while $s = s_+ > s_0$ prescribes the upper boundary $z = \eta_+(x, y, t)$ at the same latitude. The remaining parameters are defined to be positive: $a > 0$, $k > 0$, and $m > 0$ for waves with a decreasing amplitude as we ascend above the thermocline. For notational convenience, we set

$$\theta = k(q-ct).$$

The Jacobian of the map (4) is given by

$$\left(\frac{\partial(x,y,z)}{\partial(q,r,s)}\right) = \begin{pmatrix} 1 - kbe^{-ms}\cos\theta & kde^{-ms}\sin\theta & kae^{-ms}\sin\theta \\ 0 & 1 & 0 \\ mbe^{-ms}\sin\theta & mde^{-ms}\cos\theta & 1 + mae^{-ms}\cos\theta \end{pmatrix}. \tag{5}$$

The Jacobian determinant is precisely $J = 1 + (ma - kb)e^{-ms}\cos\theta - m^2a^2e^{-2ms}$ and is time independent if

$$b = \frac{ma}{k}. \tag{6}$$

Thus, the flow is volume preserving and the condition of incompressibility holds. Moreover, the amplitude of waves decreases when we ascend above the thermocline, which implies that

$$m^2a^2e^{-2ms^*} < 1, \tag{7}$$

and it also ensures, by means of the inverse function theorem, a local diffeomorphic character of (4). In this solution, the maximum amplitude of internal waves is $1/m$. The velocity and acceleration of each fluid particle are obtained by taking the derivative of (4),

$$\begin{cases} u = \frac{Dx}{Dt} = kcbe^{-ms}\cos\theta, \\ v = \frac{Dy}{Dt} = -kcde^{-ms}\sin\theta, \\ w = \frac{Dz}{Dt} = -kcae^{-ms}\sin\theta, \end{cases} \tag{8}$$

where D/Dt is the material derivative. Checking that the velocity field (8) satisfies the governing equations (3) is equivalent to obtaining a suitable pressure (cf. Ref. 29). Moreover, we get following conditions, which define the relation between the parameters of the Pollard-like solution:

$$d = -\frac{maf}{k^2c}, \quad mkc^2b + mcdf = k^2c^2a, \quad a^2 + d^2 = b^2.$$

Finally, the dynamic boundary condition (4) leads to the dispersion relation for the internal water waves

$$c^2(c^2k^2 - f^2) = (c\hat{f} + \tilde{g})^2, \tag{9}$$

where $\tilde{g} = g(\rho_+ - \rho_0)/\rho_0$ is the coefficient of reduced gravity. Choosing suitable non-dimensional variables,

$$X = c\sqrt{\frac{k}{\tilde{g}}}, \qquad \varepsilon = \frac{f}{\sqrt{\tilde{g}k}}, \qquad F = \frac{\hat{f}}{f},$$

the dispersion relation (9) can be rewritten as a polynomial equation of degree four,

$$P(X) = X^4 - \varepsilon^2(1 + F^2)X^2 - 2F\varepsilon X - 1 = 0, \tag{10}$$

with two roots, one positive and one negative,

$$X_0^+ - 1 \in (0, \varepsilon F), \qquad X_0^- + 1 \in (0, \varepsilon F),$$

for mid-latitudes $\phi \in (23°26'16'', 75°)$ on both hemispheres of earth (cf. Ref. 29). Although we solved the polynomial equation in the mid-latitudes, we note that the exact solution (4) and the dispersion relation (9) can be reduced since f vanishes in the equatorial region. In this case, both the solution and dispersion relation particularise directly to a recently derived Gerstner-like solution and its dispersion relation in the f-plane.[21]

III. MEAN FLOW PROPERTIES

In this section, we examine and describe the physical properties of the flow in the presence of the wave motion. We examine the mean velocities, the Stokes drift, and the mass flux for the internal water waves. An advantage of the transition from the Eulerian to the Lagrangian description of the flow is that we can relatively easily calculate the mean Eulerian and Lagrangian velocity. The mean Eulerian velocity is considered as the mean velocity of the flow at a fixed point, whereas the mean Lagrangian velocity is the mean velocity of a marked water particle. Moreover, the mass transport velocity arises from the mean Lagrangian velocity rather than the mean Eulerian velocity, which was noted by Stokes.[37] As a consequence, we can present the Stokes drift $\mathbf{U}^S$, which is the difference between the mean Lagrangian $\langle\mathbf{u}\rangle_L$ and the mean Eulerian $\langle\mathbf{u}\rangle_E$ velocity,[30,31]

$$\mathbf{U}^S = \langle\mathbf{u}\rangle_L - \langle\mathbf{u}\rangle_E, \tag{11}$$

where $\mathbf{u} = (u, v, w)$ is a vector with the components of the velocity in the longitudinal, latitudinal, and vertical direction, respectively. The mean flow properties are considered as three-dimensional since the particle paths are circles slightly tilted toward the equator and the particle motion is purely three-dimensional.

A. Lagrangian mean velocity

From the exact solution (4), we calculated the flow velocity in the Lagrangian setting. The explicit velocity in terms of the Lagrangian labelling variables is used to find the mean Lagrangian flow. Taking the average of the velocity over the period $T = \frac{L}{c}$, we obtain the zonal mean Lagrangian velocity

$$\langle u\rangle_L = \frac{1}{T}\int_0^T u(q-ct,r,s)\,dt = \frac{1}{T}\int_0^T kcbe^{-ms}\cos\theta dt = 0,$$

next we get the meridional mean velocity

$$\langle v\rangle_L = \frac{1}{T}\int_0^T v(q-ct,r,s)\,dt = -\frac{1}{T}\int_0^T kcde^{-ms}\sin\theta dt = 0,$$

and, finally, the vertical mean velocity

$$\langle w\rangle_L = \frac{1}{T}\int_0^T w(q-ct,r,s)\,dt = -\frac{1}{T}\int_0^T kcae^{-ms}\sin\theta dt = 0.$$

As the integral of the trigonometric functions over the wave period is equal to zero, all of the components of the mean Lagrangian velocity are zero.

B. Eulerian mean velocity

The mean Eulerian velocity is computed by taking the average of the velocity over the wave period at any fixed depth. First we fix a depth z_0 over the crest of the thermocline and under the trough of the upper boundary $z^c < z_0 < z_+^t$, where z^c represents the level of crest of the thermocline $z = \eta(x, y, t)$ and z_+^t represents the level of trough of the boundary $z = \eta_+(x, y, t)$. The crests and troughs of the thermocline are given by

$$z^c = s_0 + ae^{-ms_0}, \qquad z^t = s_0 - ae^{-ms_0},$$

and the crests and troughs of the interface $z = \eta_+(x, y, t)$ between the layers $\mathcal{M}(t)$ and $\mathcal{L}(t)$ are

$$z_+^c = s_+ + ae^{-ms_+}, \qquad z_+^t = s_+ - ae^{-ms_+},$$

respectively. The fixed depth z_0 is characterised by

$$z_0 = s - ae^{-ms}\cos\theta, \tag{12}$$

where $s = S(z_0, q, t)$. Taking the derivative with respect to the variable q of Eq. (12), we get

$$S_q = -\frac{kae^{-ms}\sin\theta}{1 + mae^{-ms}\cos\theta}.$$

We introduce the following equation in order to obtain the zonal mean Eulerian velocity:

$$\begin{aligned}
c + \langle u\rangle_E(z_0) &= \frac{1}{T}\int_0^T [c + u(x - ct, y, z_0)]\,dt = \frac{1}{L}\int_0^L [c + u(x - ct, y, z_0)]\,dx = \\
&= \frac{1}{L}\int_0^L [c + u(q - ct, z_0)]\frac{\partial x}{\partial q}\,dq = \\
&= \frac{1}{L}\int_0^L (c + kcbe^{-ms}\cos\theta)\frac{1 - a^2m^2e^{-2ms}}{1 + mae^{-ms}\cos\theta}\,dq = \\
&= c - \frac{m^2a^2c}{L}\int_0^L e^{-2mS}\,dq.
\end{aligned}$$

It implies that the zonal mean Eulerian velocity is

$$\langle u\rangle_E(z_0) = -\frac{m^2a^2c}{L}\int_0^L e^{-2mS}dq,$$

indicating a nonuniform wave-induced current, whereas $\langle u\rangle_E(z_0) \in (-m^2a^2c, 0)$ for $c > 0$ and $\langle u\rangle_E(z_0) \in (0, m^2a^2c)$ for $c < 0$. The Pollard-like solution in the equatorial region particularises to the Gerstner-like solution since f vanishes in the equatorial region. Therefore, substituting the parameters for the equatorial region and the critical value of the amplitude parameter $a = 1/m$ into the zonal Eulerian velocity, the zonal mean Eulerian velocity recalls the mean velocity for the equatorial waves in the f-plane.[21]

Subsequently, we can compute the meridional mean Eulerian velocity

$$\begin{aligned}
\langle v\rangle_E(z_0) &= \frac{1}{T}\int_0^T v(x - ct, y, z_0)\,dt = \\
&= \frac{1}{L}\int_0^L v(x - ct, y, z_0)\,dx = \frac{1}{L}\int_0^L v(q - ct, z_0)\frac{\partial x}{\partial q}\,dq = \\
&= \frac{1}{L}\int_0^L -kcde^{-mS}\sin\theta\frac{1 - m^2a^2e^{-2mS}}{1 + mae^{-mS}\cos\theta}\,dq = \\
&= \frac{fma}{kL}\int_0^L e^{-mS}\sin\theta\frac{1 - m^2a^2e^{-2mS}}{1 + mae^{-mS}\cos\theta}\,dq.
\end{aligned}$$

We can estimate the meridional mean Eulerian velocity as

$$|\langle v\rangle_E(z_0)| < \frac{fma}{kL}\int_0^L e^{-mS}(1 + mae^{-mS})\,dq,$$

keeping in mind that f changes sign across the equator. Moreover, for the equatorial internal water waves, the Coriolis parameter is equal to zero ($f = 0$) and the meridional mean Eulerian velocity vanishes. As we know, at the equator, orbits of particles are vertical and the particles do not move in the meridional direction,[6,21] which confirms the absence of the meridional mean velocity.

Following the steps above, we can find the vertical mean Eulerian velocity

$$\begin{aligned}
\langle w\rangle_E(z_0) &= \frac{1}{T}\int_0^T w(x - ct, y, z_0)\,dt = \\
&= \frac{1}{L}\int_0^L w(x - ct, y, z_0)\,dx = \frac{1}{L}\int_0^L w(q - ct, z_0)\frac{\partial x}{\partial q}\,dq = \\
&= \frac{1}{L}\int_0^L -kcae^{-mS}\sin\theta\frac{1 - m^2a^2e^{-2mS}}{1 + mae^{-mS}\cos\theta}\,dq = \\
&= -\frac{kca}{L}\int_0^L e^{-mS}\sin\theta\frac{1 - m^2a^2e^{-2mS}}{1 + mae^{-mS}\cos\theta}\,dq,
\end{aligned}$$

where it is estimated to be

$$|\langle w\rangle_E(z_0)| < \frac{kca}{L}\int_0^L e^{-mS}(1+mae^{-mS})\,dq.$$

However, we have to be aware that phasespeed c can be negative or positive in the case of Pollard-like solutions in the f-plane approximation.

C. Stokes drift

Finally, we can compute the Stokes drift in the longitudinal, meridional, and vertical direction. From the definition (11), we obtain the zonal Stokes drift

$$U^S(z_0) = \frac{m^2a^2c}{L}\int_0^L e^{-2mS}dq,$$

the meridional Stokes drift

$$V^S(z_0) = -\frac{fma}{kL}\int_0^L e^{-mS}\sin\theta\frac{1-m^2a^2e^{-2mS}}{1+mae^{-mS}\cos\theta}dq,$$

and the vertical Stokes drift

$$W^S(z_0) = \frac{kca}{L}\int_0^L e^{-mS}\sin\theta\frac{1-m^2a^2e^{-2mS}}{1+mae^{-mS}\cos\theta}dq.$$

The meridional Stokes drift is zero for the equatorially trapped internal water waves since $f = 0$ at the equator. The zonal Stokes drift in the case of the Pollard-like solution is a general formula of the Stokes drift obtained in Ref. 21. Considering the critical value of the amplitude parameter a at the equator, which is $a = 1/m$, the zonal Stokes drift reduces to the Stokes drift for the equatorial internal water waves in the f-plane.[21]

D. Mass flux

To conclude the paper, we calculate the mass flux in the layer $\mathcal{M}(t)$. An advantage of this model is that we consider the mass flux in the restricted layer $\mathcal{M}(t)$ which indicates that the mass flux is finite. As the motion of water particles is three-dimensional, we consider the flux through three planes. The mass flux through any plane S is defined as[33]

$$m = \int_S \rho\mathbf{u}\cdot\mathbf{n}dS,$$

where $\mathbf{n}$ is the normal vector to the surface S.

1. Zonal mass flux

We begin with the mass flux in the longitudinal direction m^{zonal}. We fix at the point $x = x_0$ the plane $S = [\eta, \eta_+]\times[y_1, y_2]$, where $y_1 < y_2$. The zonal mass flux is given by

$$m^{zonal} = \iint\limits_{[\eta,\eta_+]\times[y_1,y_2]} u(x_0-ct, y, z)dydz = \int_{s_0}^{s_+}\int_{-r_0}^{r_0} u\begin{vmatrix}\frac{\partial y}{\partial r} & \frac{\partial y}{\partial s}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial s}\end{vmatrix}drds, \tag{13}$$

after changing to the Lagrangian labelling variables. The fixed point $x = x_0$ implies the functional relationship $q = \gamma(x_0, s, t)$. Taking the derivative of

$$x_0 = q - be^{-ms}\sin\theta,$$

with respect to the variable s, we obtain

$$\gamma_s = -\frac{bme^{-ms}\sin\theta}{1-mae^{-ms}\cos\theta}. \tag{14}$$

From (4), we have $\frac{\partial y}{\partial r} = 1$ and $\frac{\partial z}{\partial r} = 0$, which implies that the determinant of the Jacobian in (13) is exactly $J = \frac{\partial z}{\partial s}$. Using (14), we can easily find

$$\frac{\partial z}{\partial s} = \frac{1 - m^2 a^2 e^{-2ms}}{1 - mae^{-ms}\cos\theta}.$$

Therefore, the zonal mass flux per unit of width is precisely

$$\begin{aligned} m^{zonal} &= \int_{s_0}^{s_+} kcbe^{-ms}\cos\theta \frac{1 - m^2 a^2 e^{-2ms}}{1 - mae^{-ms}\cos\theta} ds = \\ &= cma \int_{s_0}^{s_+} e^{-ms}\cos\theta \frac{1 - m^2 a^2 e^{-2ms}}{1 - mae^{-ms}\cos\theta} ds. \end{aligned} \tag{15}$$

The example of an instantaneous zonal mass flux is presented in Fig. 3. From the explicit form of the mass flux, we can conclude that the zonal mass flux stays unchanged in terms of the Northern or Southern Hemisphere since it is independent of the Coriolis parameter f, but depends on the wave phasespeed c. If we consider the parameter of amplitude $a = 1/m$ at the equator, the zonal mass flux (15) reassembles the expression of the mass flux obtained in Ref. 21 for the equatorial internal waves in the f-plane. Moreover, we can show that the average mass flux over the wave period is equal to zero. From the fixed point $x = x_0$, we get

$$\gamma_t = -\frac{mace^{-ms}\cos\theta}{1 - ame^{-ms}\cos\theta},$$

for the T-periodic function $t \mapsto \gamma(x_0, s, t)$. Therefore, the time average of the mass flux over the wave period T is zero, which is expected since the mean Lagrangian velocity is zero $\langle u \rangle_L = 0$.

2. Meridional mass flux

In the next step, we compute the meridional mass flux by fixing at the point $y = y_0$ the plane $S = [\eta, \eta_+] \times [0, L]$. From Sec. III D, we have

$$m^{meridional} = \iint_{[\eta,\eta_+]\times[0,L]} v(x - ct, y_0, z)dxdz = \int_{s_0}^{s_+}\int_0^L v \begin{vmatrix} \frac{\partial x}{\partial q} & \frac{\partial x}{\partial s} \\ \frac{\partial z}{\partial q} & \frac{\partial z}{\partial s} \end{vmatrix} dqds,$$

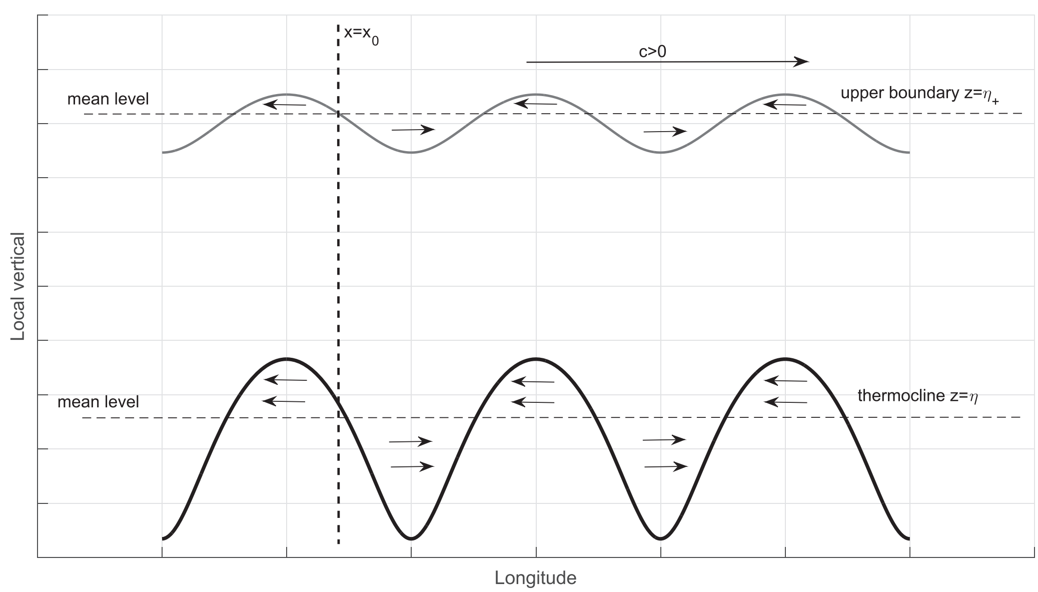

FIG. 3. The instantaneous zonal mass flux for the eastward propagating wave $c > 0$. The direction of the mass flux in this case is negative for the crest and positive for the trough of wave since $\cos\theta = \mp 1$, respectively. It is independent of the earth's hemisphere; however, the direction of mass flux is reversed if we consider $c < 0$.

in terms of the Lagrangian labelling variables. The variables x, z of (4) are independent of r; therefore, without making any assumption of the functional relationship in $y = y_0$, we obtain

$$\begin{vmatrix} \frac{\partial x}{\partial q} & \frac{\partial x}{\partial s} \\ \frac{\partial z}{\partial q} & \frac{\partial z}{\partial s} \end{vmatrix} = \begin{vmatrix} 1 - mae^{-ms}\cos\theta & mbe^{-ms}\sin\theta \\ kae^{-ms}\sin\theta & 1 + mae^{-ms}\cos\theta \end{vmatrix} = 1 - m^2a^2e^{-2ms}.$$

Thus the meridional mass flux per unit length is

$$\begin{aligned} m^{meridional} &= \frac{1}{L}\int_{s_0}^{s_+}\int_0^L -kcde^{-ms}\sin\theta(1 - m^2a^2e^{-2ms})dqds = \\ &= \frac{fma}{kL}\int_{s_0}^{s_+}\int_0^L e^{-ms}\sin\theta(1 - m^2a^2e^{-2ms})dqds. \end{aligned} \tag{16}$$

The average of the meridional mass flux over the wave period T is equal to zero. The integrand in (16) depends on the sine function and $\int_0^T \sin\theta dt = 0$. The vanishing meridional mass flux over the wave period is a natural result since the meridional mean velocity is zero $\langle v\rangle_L = 0$. The meridional mass flux is independent of the direction of wave propagation c; however, it depends on the hemisphere of earth. An instantaneous meridional mass flux is depicted in Fig. 4. Furthermore, for the equatorial internal water wave, $f = 0$ and the meridional mass flux is zero all times. This is reasonable because for the equatorial internal waves, the particle orbits are vertical and there is no motion in the meridional direction indicating any meridional mean Lagrangian velocity.

3. Vertical mass flux

Finally, we can present the vertical mass flux. We fix a point $z^c < z_0 < z_+^t$ between the crest of the thermocline z^c and the trough z_+^t of the interface between $\mathcal{M}(t)$ and $\mathcal{L}(t)$,

$$z_0 = s - ae^{-ms}\cos\theta, \text{ where } s = \beta(z_0, q, t). \tag{17}$$

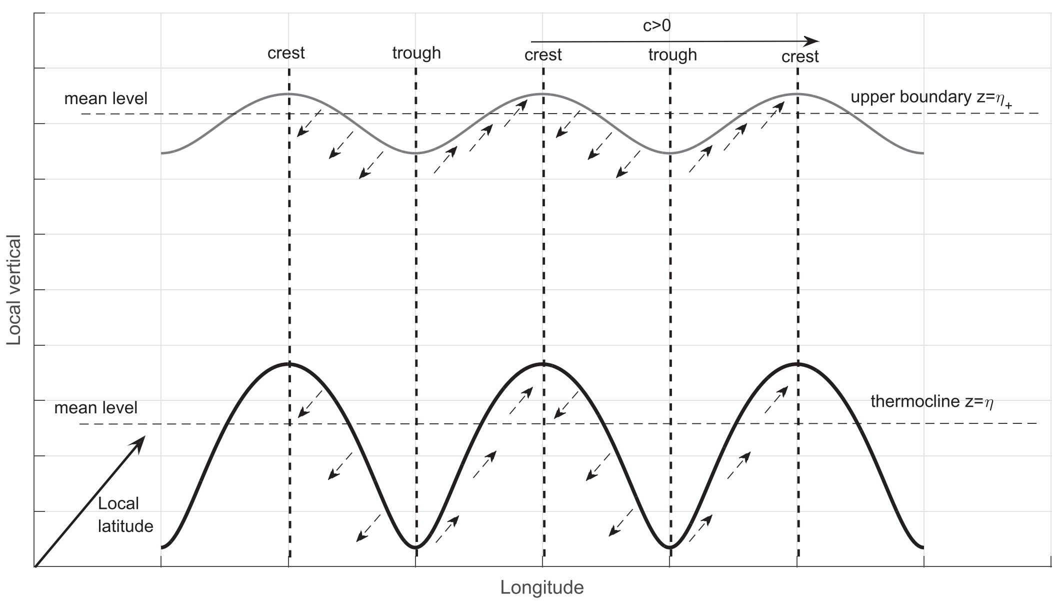

FIG. 4. The schematic of the instantaneous meridional mass flux for the eastward propagating wave $c > 0$ on the Northern Hemisphere. For the crest and troughs of the wave, the meridional mass flux vanishes since $\sin\theta = 0$ at those points. If we consider this scenario on the Southern Hemisphere, the direction of the instantaneous mass flux is reversed. However, the meridional mass flux stays unchanged if we take the westward propagating wave $c < 0$.

At the point $z = z_0$, we fix the plane $S = [0, L] \times [y_1, y_2]$, and the vertical mass flux in the Lagrangian labelling variable is

$$m^{vertical} = \iint_{[0,L]\times[y_1,y_2]} w(x - ct, y, z_0)dxdy = \int_0^L \int_{-r_0 1}^{-r_0} w \begin{vmatrix} \frac{\partial x}{\partial q} & \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial q} & \frac{\partial y}{\partial r} \end{vmatrix} dqdr.$$

Since $\frac{\partial x}{\partial r} = 0$ and $\frac{\partial y}{\partial r} = 1$, we have that the Jacobian determinant of the transformation is precisely $J = \frac{\partial x}{\partial q}$. Taking the derivative of (17) with respect to q, we get

$$\beta_q = -\frac{kae^{-m\beta} \sin\theta}{1 + mae^{-m\beta}\cos\theta},$$

which yields

$$\frac{\partial x}{\partial q} = \frac{1 - m^2a^2e^{-2m\beta}}{1 + mae^{-m\beta}\cos\theta}.$$

Following the above calculations, we obtain the vertical mass flux per unit width

$$m^{vertical} = -kca \int_0^L e^{-m\beta} \sin\theta \frac{1 - m^2a^2e^{-2m\beta}}{1 + mae^{-m\beta}\cos\theta} dq. \tag{18}$$

Figure 5 presents the schematic of the instantaneous vertical mass flux. The direction of the vertical mass flux depends on the direction of the wave propagation, but is independent of the earth's hemisphere.

Furthermore, we obtain

$$\beta_t = \frac{kcae^{-m\beta} \sin\theta}{1 + mae^{-m\beta}\cos\theta}$$

for the T-periodic function $t \mapsto \beta(q, t, z_0)$. If we consider β_t, we can see that the time average of the vertical mass flux over the wave period is zero. We proved that the vertical mean Lagrangian velocity is zero $\langle w \rangle_L = 0$. Therefore, it is reasonable that the total vertical mass flux over the wave period vanishes. In conclusion, we have proved that the Pollard-like internal wave solution has no net wave transport over the wave period.

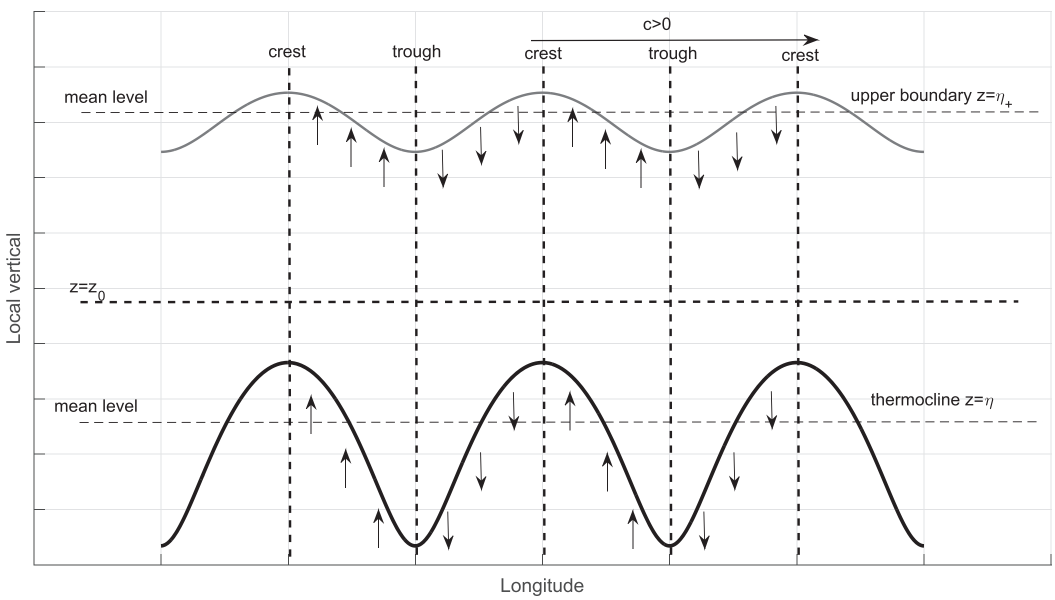

FIG. 5. The instantaneous vertical mass flux for the eastward propagating wave $c > 0$. For the westward propagating wave, the state is reversed although it is unaltered after change of the earth's hemisphere. The instantaneous mass flux is zero for the crests and troughs of the wave since $\sin\theta = 0$ at those points.

IV. DISCUSSION

In this paper, we considered a new Pollard-like internal water wave solution which is a remarkable extension of the Pollard solution[32] to describe nonlinear internal water waves, whereas the Pollard solution is a modification of the Gerstner solution[13] to vertically stratified fluids in a rotating system. The wave in the Pollard-like solution experiences a cross-wave tilt and particles move in planes slightly tilted to the vertical,[10,32] whereas in Gerstner-like solutions,[3,13,21] the path of particles is contained in parallel vertical planes. We analyzed the mean flow velocities of the flow which are three-dimensional vectors since the Pollard-like solution represents a fully three-dimensional periodic zonally traveling wave, in contrast to the mean velocities derived for the equatorially trapped waves.[20,36] In papers,[6,20,21,36] the mean velocities and mass flux are presented only in the zonal direction. Although in Refs. 6, 20, and 36 the three-dimensional effect is captured by considering the decay of the amplitude of waves in the meridional direction, the solution represents a two-dimensional wave, and therefore the physical properties are considered only in the zonal direction. However, two-dimensional mean velocities and mass transport were presented for the equatorial flows in the f-plane in the case of solutions including a transverse variable current.[28,34] Recent studies considered Gerstner-like solutions in a region close to the equator,[6,7,21,28,34] whereas a study here considers the Pollard-like solution, which is not only three-dimensional without underlying currents but also valid for mid-latitudes, which is a significant improvement and extension of exact and explicit solutions for nonlinear geophysical flows.

Summarising the results, we proved that the mean Lagrangian velocities over a wave period are equal to zero. We calculated the Stokes drift, and although there exists a hallmark of the Stokes drift, it is balanced by the mean Eulerian velocity. According to Ref. 31, the mean Lagrangian velocity is sometimes called mass-transport velocity, and it represents somewhat the mass transport in oceans. Arguing along this line, we showed that the mass flux vanishes over a wave period resulting in a zero net wave transport, which is expected since the mean Lagrangian velocities are zero.

A quantitative discussion can be done as follow. Let us consider the latitude $\phi = 45^\circ\, N$ and the difference in the density $\Delta\rho/\rho = 6 \times 10^{-3}$. The wave of the wavelength $L = 100$ m is associated with the wavenumber $k \approx 0.0628$ m^{-1} and yields the phasespeed in the considered model $c \approx -0.9671$ m/s or $c \approx 0.9687$ m/s, whereas the phasespeed of a standard Gerstner deep-water wave is $c \approx 0.9679$ m/s. The period of the considered wave is $T \approx 103.09$ s. Given the phasespeed and wavenumber, we can calculate $m \approx 0.0622$ m^{-1}, which gives the maximal amplitude of the wave to be about 16.0772 m. Setting the amplitude of the wave at the thermocline to be 10 m and the average $s_+ - s_0 = 50$ m, we get that the oscillation of the upper boundary of the layer $\mathcal{M}(t)$ is $ae^{-m(s_0+50)} \approx 0.4$ m, whereas at 100 m above the thermocline, the oscillation is about 0.02 m (Fig. 6). Considering the wave motion extends undisturbed to the ocean's surface, the oscillation of the free surface caused by the

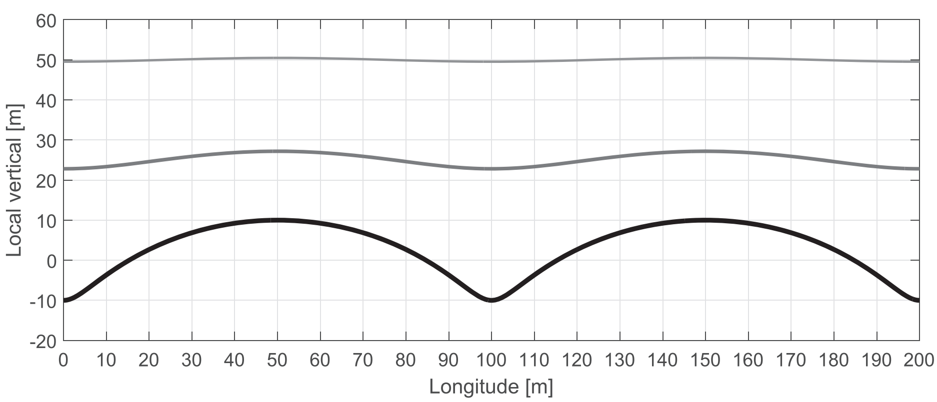

FIG. 6. Schematic showing the internal wave amplitude. The wave propagating at the thermocline is set to have the wavelength $L = 100$ m and $k = 6.28 \times 10^{-2}$ m^{-1}. The wave propagates at a latitude $\phi = 45^\circ\, N$ with $\Delta\rho/\rho = 6 \times 10^{-3}$. Taking the amplitude of the oscillation of the thermocline to be 10 m, the amplitude of the wave at the height 25 m and 50 m above the thermocline is 2.11 m and 0.4 m, respectively.

internal water waves is indistinguishable from the common ocean's surface perturbation because of the exponential decrease of the amplitude of the internal waves.

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